Self-consistent perturbation series for stationary homogeneous turbulence

1969 J. Phys. A: Gen. Phys. 2 181

(http://iopscience.iop.org/0022-3689/2/2/005)

J. PHYS. A (GEN. PHYS.), 1969, SER. 2, VOL. 2. PRINTED IN GREAT BRITAIN

Self-consistent perturbation series for stationary homogeneous turbulence

R. PHYTHIAN
Department of Physics, University College of Swansea
MS. received 21st August 1968

Abstract. A self-consistent perturbation procedure is developed for the problem of stationary homogeneous turbulence of an incompressible fluid. It differs from the theories of Herring and Edwards in that the direct-interaction approximation is obtained in the first non-trivial approximation. It provides a simple means of obtaining equations relating the two-velocity correlation function and response function without necessitating the analysis of diagrams, representing terms of a perturbation series, of large order as required in the formalism of Wyld and Lee. The relation to the work of the authors mentioned is briefly discussed.

1. Introduction

A systematic use of the perturbation theory familiar in quantum mechanics was introduced into the problem of hydrodynamic turbulence by Wyld (1961). The zeroth-order term of this perturbation series describes the fluid in the absence of the non-linear terms of the Navier–Stokes equation. The disadvantage of this approach is that it is necessary to carry out a rather elaborate analysis of the 'Feynman' diagrams, representing the higher-order terms of the series, in order to obtain equations relating quantities of physical interest, such as the two-velocity correlation function and the response function. The analysis is similar to that involved in deriving the Dyson equations of quantum electrodynamics but is more complicated. In fact, the equations given in Wyld's paper are incorrect for the actual turbulence problem, although valid for a simplified model system, as was pointed out by Lee (1965). Different equations have been given by Lee which have been checked to reproduce the perturbation series correctly up to sixth order. The direct-interaction approximation of Kraichnan may be obtained from these theories by retaining only the simplest terms in the equations.

An alternative approach, introduced by Edwards (1964), bases a perturbation expansion on a zeroth-order term which allows approximately for the transfer of energy by the non-linear terms of the Navier–Stokes equation and which is determined self-consistently. In his paper, Edwards derives two different series: the first of these is for the probability density of the velocity field at a given time, the second is for the probability density of the velocity throughout all time. In deriving the former, use is made of the fact that a closed functional-differential equation for the probability density can be written down when the external 'stirring' forces have a Gaussian distribution with delta-function correlations in time (see also Novikov (1965)). In the series obtained, the external parameters, i.e. the viscosity and the correlation function of the external forces, appear only in the zeroth-order terms. This property does not seem to be shared by the second expansion. The derivation of higher-order terms proves to be very tedious in this approach.

A somewhat similar method is employed by Herring (1965), again for the case of the probability distribution at a particular time, and with the assumption that the applied forces have a Gaussian distribution with delta-function time correlations. The series obtained differs from that of Edwards.

It is interesting to note that neither of these self-consistent expansions yields the direct-interaction approximation of Kraichnan (which has received some experimental support) and in this respect they differ from the original perturbation theory of Wyld.

In the present work we introduce a self-consistent perturbation procedure which does yield the direct-interaction approximation as the simplest non-trivial approximation. Instead of deriving an expansion directly for the probability distribution as described above we proceed by writing down a series solution for the Navier–Stokes equation.

This is based on a zeroth-order equation which is different for each realization of the external force field. The zeroth-order equation for the velocity field contains (i) a modified viscous dissipation term allowing for the transfer of energy from each Fourier mode by inertial as well as by viscous forces, and (ii) a random force, which differs from the external force, to allow for the transfer of energy to each mode by the inertial forces as well as by the external force. These two quantities are chosen so that, on averaging over all realizations of the Gaussian external force, the zeroth-order equation gives the exact two-velocity correlation function and response function.

We finally obtain the correlation function and response function expressed as infinite series, each term of which except the first contains only these same quantities. The zeroth-order terms involve the viscosity and the correlation function of the external force. The direct-interaction approximation is obtained by making the simplest non-trivial truncation of these series. The formalism as described above differs from that of Wyld in that no 'dressed' vertex appears, but this may be incorporated in a straightforward manner as described later.

2. Derivation of the expansion

2.1. *Preliminary considerations*

In order to keep the notation as simple as possible we shall present the argument for the case of the Burgers model equation. The whole procedure goes through in just the same way for the Navier–Stokes equation when the pressure term has been eliminated in the usual way. The 'velocity' field $v(x, t)$ satisfies the equation

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} - v \frac{\partial v}{\partial x} + f(x, t)$$

where $f(x, t)$ is an applied random 'force' field which is statistically homogeneous and stationary and has a Gaussian distribution with zero mean.

We are interested in the resulting homogeneous stationary statistical distribution of the velocity field. It is convenient to consider the 'fluid' enclosed in a space–time box of volume VT and satisfying periodic boundary conditions so that the usual Fourier decompositions may be introduced:

$$v(x, t) = \frac{1}{VT} \sum_{k,\omega} v(k, \omega) \exp(\mathrm{i}kx + \mathrm{i}\omega t)$$

$$f(x, t) = \frac{1}{VT} \sum_{k,\omega} f(k, \omega) \exp(\mathrm{i}kx + \mathrm{i}\omega t)$$

where the summation extends over all wave numbers (k, ω) of the form $(2\pi n/V, 2\pi m/T)$, where n, m are integers. The Fourier transforms must satisfy the reality conditions

$$v^*(k, \omega) = v(-k, -\omega)$$

$$f^*(k, \omega) = f(-k, -\omega).$$

We also assume that, in each realization, the centre of mass of the fluid remains at rest so that $v(0, \omega)$ and $f(0, \omega)$ are both zero. In the limit as V and T tend to infinity, we have

$$\frac{1}{VT} \sum_{k,\omega} \to \frac{1}{(2\pi)^2} \int \mathrm{d}k \int \mathrm{d}\omega.$$

The Fourier components are seen to satisfy the equation

$$\mathrm{i}\omega v(k, \omega) = -\nu k^2 v(k, \omega) + \frac{\lambda}{VT} \sum M(k, \omega; k_1, \omega_1, k_2, \omega_2) v(k_1, \omega_1) v(k_2, \omega_2) + f(k, \omega) \quad (1)$$

where

$$M(k, \omega; k_1, \omega_1, k_2, \omega_2) = -\tfrac{1}{2}\mathrm{i}k\delta_{k_1+k_2,k}\delta_{\omega_1+\omega_2,\omega}$$

and an expansion parameter λ has been introduced for future convenience.

Homogeneity and stationariness lead to the following conditions on the expectation values of products of the Fourier transforms

$$\begin{aligned} \langle f(k_1, \omega_1) \ldots f(k_n, \omega_n) \rangle &= 0 \\ \langle v(k_1, \omega_1) \ldots v(k_n, \omega_n) \rangle &= 0 \end{aligned} \quad \text{unless} \begin{cases} k_1 + \ldots k_n = 0 \\ \omega_1 + \ldots \omega_n = 0 \end{cases}$$

$$\langle f(k, \omega) f(k', \omega') \rangle = VT\delta_{k+k',0}\delta_{\omega+\omega',0}h(k, \omega)$$

$$\langle v(k, \omega) v(k', \omega') \rangle = VT\delta_{k+k',0}\delta_{\omega+\omega',0}U(k, \omega).$$

This is in accord with the usual definitions of the spectral functions $h(k, \omega)$, $U(k, \omega)$ in the limit as $V, T \to \infty$

$$\langle f(x, t) f(x', t') \rangle = \frac{1}{(2\pi)^2} \int \mathrm{d}k \int \mathrm{d}\omega\, h(k, \omega) \exp\{\mathrm{i}k(x-x') + \mathrm{i}\omega(t-t')\}$$

$$\langle v(x, t | v(x', t') \rangle = \frac{1}{(2\pi)^2} \int \mathrm{d}k \int \mathrm{d}\omega\, U(k, \omega) \exp\{\mathrm{i}k(x-x') + \mathrm{i}\omega(t-t')\}.$$

The Gaussian nature of the distribution of f leads to the pairing property by means of which expectation values of an even number of the f can be written as the sum of all possible pairings, for example

$$\begin{aligned} \langle f(k_1, \omega_1) f(k_2, \omega_2) f(k_3, \omega_3) f(k_4, \omega_4) \rangle = {} & \langle f(k_1, \omega_1) f(k_2, \omega_2) \rangle \langle f(k_3, \omega_3) f(k_4, \omega_4) \rangle \\ & + \langle f(k_1, \omega_1) f(k_3, \omega_3) \rangle \langle f(k_2, \omega_2) f(k_4, \omega_4) \rangle \\ & + \langle f(k_1, \omega_1) f(k_4, \omega_4) \rangle \langle f(k_2, \omega_2) f(k_3, \omega_3) \rangle \end{aligned}$$

while the expectation value of the product of an odd number of the f is zero.

2.2. *Derivation of the perturbation series for the velocity*

The expansion of Wyld is obtained by iterating equation (1) with the zeroth-order term given by setting $\lambda = 0$. Instead of following this approach, we rewrite the equation in the form

$$\begin{aligned} \mathrm{i}\omega v(k, \omega) = {} & \alpha(k, \omega) v(k, \omega) + g(k, \omega) + R(k, \omega) v(k, \omega) + e(k, \omega) \\ & + \frac{\lambda}{VT} \sum M(k, \omega; k_1, \omega_1, k_2, \omega_2) v(k_1, \omega_1) v(k_2, \omega_2) \end{aligned} \tag{2}$$

where

$$\alpha(k, \omega) + R(k, \omega) = -\nu k^2$$

$$g(k, \omega) + e(k, \omega) = f(k, \omega).$$

The function $g(k, \omega)$ is a random function which is functionally dependent on $f(k, \omega)$. The quantity $\alpha(k, \omega)$ is not a random function, i.e. it is the same for each realization of the force field; it does, however, depend on the statistical distribution of f. In addition, both g and α satisfy the usual reality conditions.

The zeroth-order equation on which the perturbation expansion is based is

$$\mathrm{i}\omega v(k, \omega) = \alpha(k, \omega) v(k, \omega) + g(k, \omega) \tag{3}$$

where the quantities α and g will eventually be chosen in such a way that certain statistical properties of the velocity field are the same for equation (3) as for the full equation (1). Clearly, it is possible in principle to choose them so that the equations are identical, but since this is impracticable we shall be content with imposing the conditions that the two-velocity correlation function and the response function should be the same for these two equations.

When λ is zero, the obvious choice is made

$$\alpha(k, \omega) = -\nu k^2$$

$$g(k, \omega) = f(k, \omega)$$

so that equations (1) and (3) are identical. This means that R and e vanish when λ is zero, and we shall assume that they may be expanded in powers of λ, whence

$$R(k,\omega) = \lambda R_1(k,\omega) + \lambda^2 R_2(k,\omega) + \ldots$$
$$e(k,\omega) = \lambda e_1(k,\omega) + \lambda^2 e_2(k,\omega) + \ldots .$$

Starting with the zeroth-order approximation for $v(k,\omega)$ given by equation (3), namely $g(k,\omega)\{\mathrm{i}\omega - \alpha(k,\omega)\}^{-1}$, and using the series for R and e above, we can express $v(k,\omega)$ as a series in increasing powers of λ:

$$\begin{aligned}
v(k,\omega) = {} & \frac{g(k,\omega)}{\mathrm{i}\omega - \alpha(k,\omega)} \\
& + \lambda\left[\frac{R_1(k,\omega)g(k,\omega)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}^2} + \frac{e_1(k,\omega)}{\mathrm{i}\omega - \alpha(k,\omega)}\right. \\
& \left. + \frac{1}{VT}\sum \frac{M(k,\omega;k_1,\omega_1,k_2,\omega_2)g(k_1,\omega_1)g(k_2,\omega_2)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}\{\mathrm{i}\omega_1 - \alpha(k_1,\omega_1)\}\{\mathrm{i}\omega_2 - \alpha(k_2,\omega_2)\}}\right] \\
& + \lambda^2\left(\frac{R_2(k,\omega)g(k,\omega)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}^2} + \frac{e_2(k,\omega)}{\mathrm{i}\omega - \alpha(k,\omega)} + \frac{R_1^2(k,\omega)g(k,\omega)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}^3}\right. \\
& + \frac{R_1(k,\omega)e_1(k,\omega)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}^2} + \frac{1}{VT}\sum \frac{R_1(k,\omega)M(k,\omega;k_1,\omega_1,k_2,\omega_2)g(k_1,\omega_1)g(k_2,\omega_2)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}^2\{\mathrm{i}\omega_1 - \alpha(k_1,\omega_1)\}\{\mathrm{i}\omega_2 - \alpha(k_2,\omega_2)\}} \\
& + \frac{2}{VT}\sum \frac{M(k,\omega;k_1,\omega_1,k_2,\omega_2)R_1(k_1,\omega_1)g(k_1,\omega_1)g(k_2,\omega_2)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}\{\mathrm{i}\omega_1 - \alpha(k_1,\omega_1)\}^2\{\mathrm{i}\omega_2 - \alpha(k_2,\omega_2)\}} \\
& + \frac{2}{VT}\sum \frac{M(k,\omega;k_1,\omega_1,k_2,\omega_2)e_1(k_1,\omega_1)g(k_2,\omega_2)}{\{\mathrm{i}\omega - \alpha(k,\omega)\}\{\mathrm{i}\omega_1 - \alpha(k_1,\omega_1)\}\{\mathrm{i}\omega_2 - \alpha(k_2,\omega_2)\}} \\
& + \frac{2}{V^2T^2}\sum M(k,\omega;k_1,\omega_1,k_2,\omega_2)M(k_2,\omega_2;k_1',\omega_1',k_2',\omega_2')g(k_1,\omega_1)g(k_1',\omega_1') \\
& \qquad \times g(k_2',\omega_2') \\
& \qquad \times [\{\mathrm{i}\omega - \alpha(k,\omega)\}\{\mathrm{i}\omega_1 - \alpha(k_1,\omega_1)\}\{\mathrm{i}\omega_2 - \alpha(k_2,\omega_2)\} \\
& \qquad \left. \times \{\mathrm{i}\omega_1' - \alpha(k_1',\omega_1')\}\{\mathrm{i}\omega_2' - \alpha(k_2',\omega_2')\}]^{-1}\right) \\
& + \mathrm{O}(\lambda^3). \qquad (4)
\end{aligned}$$

The terms of the series may conveniently be represented by diagrams similar to the 'tree' diagrams of Wyld. The diagrams corresponding to the terms written down in equation (4) are respectively

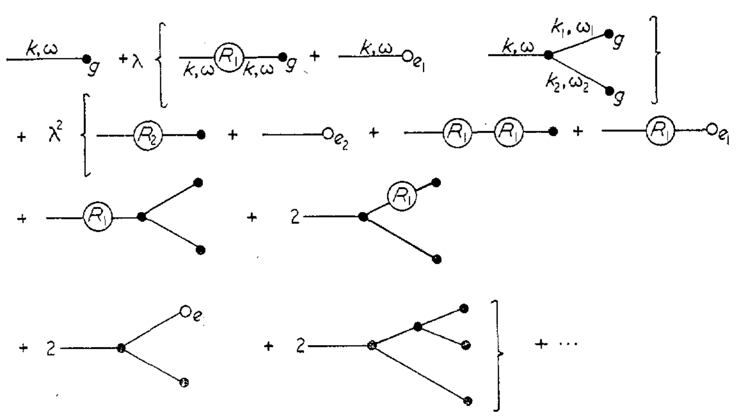

Figure 1

where the wave number labels on the lines have been omitted from the second-order diagrams for brevity. From these examples, the general rules for such diagrams will be apparent.

2.3. *Self-consistency conditions*

We now require that certain statistical properties are the same for the zeroth-order equation as for the full equation. First of all we must ensure that the random function $g(k, \omega)$ is such that the velocity field which is a solution of equation (3) has a homogeneous and stationary distribution with zero mean. This will certainly be so if we have

$$\langle g(k_1, \omega_1) \ldots g(k_n, \omega_n) \rangle = 0$$

unless

$$\begin{cases} k_1 + \ldots + k_n = 0 \\ \omega_1 + \ldots + \omega_n = 0 \end{cases}$$

and

$$g(0, \omega) = 0.$$

In addition, it transpires that we may impose the further restriction that the g have a Gaussian distribution. The spectral function is denoted by $\beta(k, \omega)$, so we have

$$\langle g(k, \omega) g(k', \omega') \rangle = VT \delta_{k+k',0} \delta_{\omega+\omega',0} \beta(k, \omega).$$

The remaining conditions are that the response function and the two-velocity correlation function are the same for the two equations. The response function is defined as follows: an infinitesimal perturbing force $\eta(k, \omega)$ is added to the right-hand side of the equation, the resulting change in the velocity being given by

$$\delta v(k, \omega) = \frac{1}{VT} \sum_{k', \omega'} S(k, \omega; k', \omega') \eta(k', \omega') + O(\eta^2).$$

The quantity S depends on the particular realization of the random function f, but, averaging over these realizations, we obtain the response function $S(k, \omega)$

$$\langle S(k, \omega; k', \omega') \rangle = VT \delta_{k,k'} \delta_{\omega,\omega'} S(k, \omega).$$

Equivalently, we may write

$$S(k, \omega) = \left\langle \frac{\partial v(k, \omega)}{\partial g(k, \omega)} \right\rangle.$$

For the zeroth-order equation the perturbed equation is

$$i\omega v_1(k, \omega) = \alpha(k, \omega) v_1(k, \omega) + g(k, \omega) + \eta(k, \omega)$$

so that

$$\delta v(k, \omega) = v_1(k, \omega) - v(k, \omega) = \frac{\eta(k, \omega)}{i\omega - \alpha(k, \omega)}$$

and, since α is not a random function, the response function is simply $\{i\omega - \alpha(k, \omega)\}^{-1}$. This will be denoted by $\Omega^{-1}(k, \omega)$. The reality condition gives

$$\Omega^*(k, \omega) = \Omega(-k, -\omega).$$

For the full equation the response function is obtained from equation (4) in the form of an infinite series

$$\frac{1}{\Omega(k, \omega)} + \lambda \frac{R_1(k, \omega)}{\Omega^2(k, \omega)} + \lambda^2 \left\{ \frac{R_1'^2(k, \omega)}{\Omega^3(k, \omega)} + \frac{R_2(k, \omega)}{\Omega^2(k, \omega)} + \frac{4}{V^2 T^2} \sum \frac{M(k, \omega; k_1, \omega_1, k_2, \omega_2) M(k_2, \omega_2; -k_1, -\omega_1, k, \omega) \beta(k_1, \omega_1)}{\Omega^2 \Omega_1 \Omega_2 \Omega_1^*} \right\}. \quad (5)$$

Here Ω_1 denotes $\Omega(k_1, \omega_1)$ etc. Many terms have dropped out because of the homogeneity conditions and the vanishing of $\langle g(k, \omega)\rangle$. We have also assumed that $\langle e_1(k, \omega)\rangle$ is zero; in fact, it follows that, since both $\langle f(k, \omega)\rangle$ and $\langle g(k, \omega)\rangle$ are zero, we have $\langle e_n(k, \omega)\rangle$ zero for all n.

The diagrammatic representation of the series is easily obtained; we simply take each diagram of figure 1 and erase one g in all possible ways (if there are no g's to erase then we get zero); we then average the remaining random functions. We thus obtain

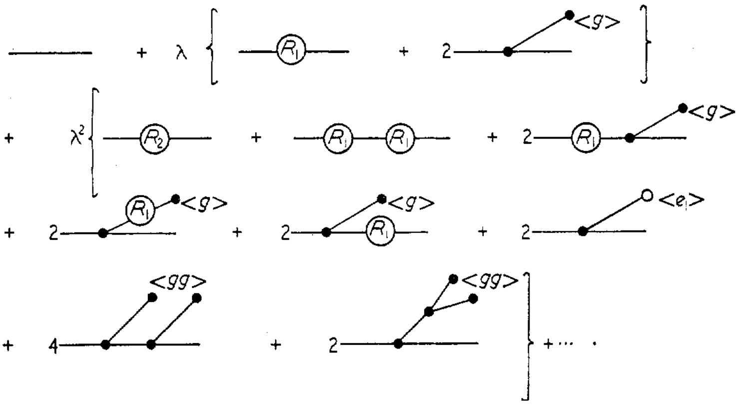

Figure 2

The diagrams involving $\langle g\rangle$ or $\langle e_1\rangle$ clearly give zero, while the last diagram shown gives zero because $M(0,0; k_1,\omega_1, k_2,\omega_2)$ vanishes.

If the zeroth-order equation gives the exact response function then the terms of order λ, λ^2, etc. in the above series must all vanish. This gives

$$R_1(k, \omega) = 0$$

$$R_2(k, \omega) = -\frac{4}{V^2T^2}\sum \frac{M(k,\omega; k_1,\omega_1, k_2,\omega_2)M(k_2,\omega_2; -k_1, -\omega_1, k,\omega)\beta(k_1, \omega_1)}{\Omega_1\Omega_1{}^*\Omega_2}$$

and equations for R_3, R_4, etc., which will be considered later.

The two-velocity correlation function is easily found for the zeroth-order equation; the corresponding spectral function is seen to be

$$\frac{\beta(k, \omega)}{\Omega(k, \omega)\Omega^*(k, \omega)}.$$

For the full equation we again obtain the result in the form of an infinite series by multiplying together two series of the form (4), one for $v(k, \omega)$ the other for $v(-k, -\omega)$, and carrying out the averaging over all realizations of the random function f. In this way we get

$$\begin{aligned}
U(k, \omega) = {} & \frac{\beta(k, \omega)}{\Omega(k, \omega)\Omega^*(k, \omega)} + \lambda\left\{\frac{1}{VT}\frac{\langle g(k, \omega)e_1(-k, -\omega)\rangle}{\Omega(k, \omega)\Omega^*(k, \omega)}\right. \\
& + \frac{1}{V^2T^2}\sum \frac{M(k,\omega; k_1,\omega_1, k_2,\omega_2)\langle g(k_1, \omega_1)g(k_2, \omega_2)g(-k, -\omega)\rangle}{\Omega\Omega^*\Omega_1\Omega_2} \\
& \left. + \begin{pmatrix}\text{similar terms with } k, \omega \\ \text{replaced by } -k, -\omega\end{pmatrix}\right\} \\
& + \lambda^2\left\{\frac{R_2(k, \omega)\beta(k, \omega)}{\Omega^2\Omega^*} + \frac{1}{VT}\frac{\langle e_2(k, \omega)g(-k, -\omega)\rangle}{\Omega\Omega^*}\right. \\
& + \frac{2}{V^3T^3}\sum M(-k, -\omega; k_1{}',\omega_1{}', k_2{}',\omega_2{}')M(k_2{}',\omega_2{}'; k_3{}',\omega_3{}', k_4{}',\omega_4{}') \\
& \quad \times \langle g(k, \omega)g(k_1{}', \omega_1{}')g(k_3{}', \omega_3{}')g(k_4{}', \omega_4{}')\rangle \\
& \quad \times (\Omega\Omega^*\Omega_{1'}\Omega_{2'}\Omega_{3'}\Omega_{4'})^{-1}
\end{aligned}$$

$$+(\text{similar terms with } k,\ \omega \text{ replaced by } -k,\ -\omega)$$
$$+\frac{1}{V^3T^3}\sum M(k,\omega;k_1,\omega_1,k_2,\omega_2)M(-k,-\omega;k_1',\omega_1',k_2',\omega_2')$$
$$\times\langle g(k_1,\omega_1)g(k_2,\omega_2)g(k_1',\omega_1')g(k_2',\omega_2')\rangle$$
$$\times(\Omega\Omega^*\Omega_1\Omega_2\Omega_{1'}\Omega_{2'})^{-1}$$
$$+\frac{1}{VT}\frac{\langle e_1(k,\omega)e_1(-k,-\omega)\rangle}{\Omega\Omega^*}$$
$$+\text{terms involving expectation values of the form } \langle gge_1\rangle\Big\}$$
$$+O(\lambda^3). \tag{6}$$

Again the diagrammatic representation of the terms is easily found.

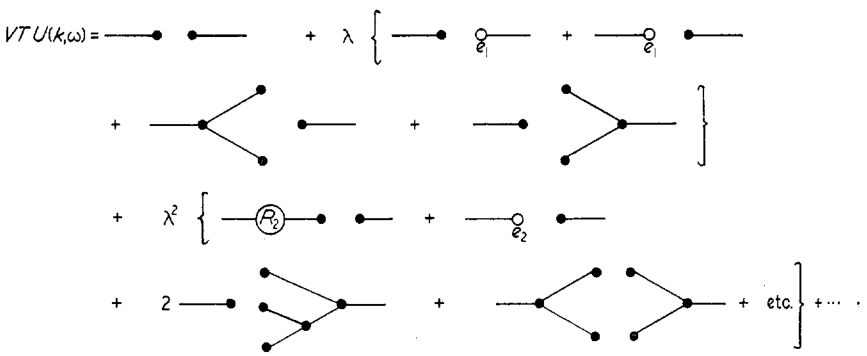

Figure 3

Here it is understood that the line on the extreme left of each diagram carries a wave number (k, ω) while that on the right has wave number $(-k, -\omega)$. It is also understood that the expectation value of the product of all the random functions appearing in a diagram is to be taken; later on a way of indicating this explicitly will be introduced.

If the zeroth-order term of this expansion gives the exact correlation function, then the coefficients of λ, λ^2, etc., must all vanish. This will give rise to equations relating expectation values of products of the functions g and e_n and also the quantities R_n. These relations do not appear to determine uniquely the functions g, e_n and we are at liberty to impose further conditions. The conditions which seem to lead to the simplest results are

$$e_n(k,\omega)=\begin{cases}0 & \text{for } n \text{ odd}\\ A_n(k,\omega)g(k,\omega) & \text{for } n \text{ even}\end{cases}$$

where $A_n(k, \omega)$ are non-random functions. This is the only case we shall consider here.

These additional conditions imply that the expectation value of any product of g and e_n may be written in terms of the functions A_n and β by using the pairing property. From this it will be seen that the equations resulting from the vanishing of the coefficient of λ^{2n} in the series (6) will determine A_{2n} in terms of A_2, A_4, ... A_{2n-2} and R_2, R_4, ... R_{2n}. The equations resulting from series (5) for the response function are seen to relate R_{2n} to A_{2n-2}, ... A_2, β, R_{2n-2}, ... R_2. We are thus able to solve the equations successively to find A_2, A_4, ...; R_2, R_4, ... in terms of α and β. (It may be seen that R_n is zero for n odd.) Actually, only the real part of $A_n(k, \omega)$ is determined but this is sufficient.

For A_2 we obtain

$$\beta(k,\omega)\{A_2(k,\omega)+A_2(-k,-\omega)\}$$
$$=-\frac{2}{VT}\sum\frac{M(k,\omega;k_1,\omega_1,k_2,\omega_2)M(-k,-\omega;\ -k_1,\ -\omega_1,\ -k_2,\ -\omega_2)\beta(k_1,\omega_1)\beta(k_2,\omega_2)}{\Omega_1\Omega_2\Omega_1^*\Omega_2^*}.$$

From this we calculate $\beta(k, \omega)$ to second order in λ since we have

$$f(k, \omega) = g(k, \omega) + \lambda^2 A_2(k, \omega) g(k, \omega) + O(\lambda^4)$$

hence

$$h(k, \omega) = \beta(k, \omega) + \lambda^2\{A_2(k, \omega) + A_2(-k, -\omega)\}\beta(k, \omega) + O(\lambda^4)$$
$$= \beta(k, \omega) - \frac{2\lambda^2}{VT}\sum M(k,\omega; k_1,\omega_1, k_2,\omega_2) M(-k, -\omega; -k_1, -\omega_1, -k_2, -\omega_2)$$
$$\times \beta(k_1, \omega_1)\beta(k_2, \omega_2) \times (\Omega_1\Omega_1{}^*\Omega_2\Omega_2{}^*)^{-1}$$
$$+ O(\lambda^4).$$

To second order in λ we have then, in the limit as $V, T \to \infty$, the following equations relating α and β

$$\alpha(k, \omega) = -\nu k^2 - \frac{\lambda^2}{(2\pi)^2}\int dk_1 \int d\omega_1 \frac{k(k-k_1)\beta(k_1, \omega_1)}{\Omega_1\Omega_1{}^*\Omega(k-k_1, \omega-\omega_1)}$$

$$\beta(k, \omega) = h(k, \omega) + \frac{\lambda^2}{2(2\pi)^2}\int dk_1 \int d\omega_1 \frac{k^2\beta(k_1, \omega_1)\beta(k-k_1, \omega-\omega_1)}{\Omega_1\Omega_1{}^*\Omega(k-k_1, \omega-\omega_1)\Omega^*(k-k_1, \omega-\omega_1)}$$

the correlation function and response function being given in terms of α and β by the relations

$$U(k, \omega) = \frac{\beta(k, \omega)}{\Omega(k, \omega)\Omega^*(k, \omega)} = \frac{\beta(k, \omega)}{\{i\omega - \alpha(k, \omega)\}\{i\omega - \alpha(k, \omega)\}^*}$$

$$S(k, \omega) = \frac{1}{\Omega(k, \omega)} = \frac{1}{i\omega - \alpha(k, \omega)}.$$

These equations are exactly the same as those of the direct-interaction approximation when expressed in terms of S and U.

We now extend the diagram technique to show that expectation values of products of random functions are to be taken. As mentioned above, since e_n differs from g only by a factor which is a non-random function, it follows immediately that the pairing rule applies to an expectation value of any product of the g and the e_n. For example

$$\langle e_2(k_1, \omega_1)e_4(k_2, \omega_2)g(k_3, \omega_3)g(k_4, \omega_4)\rangle$$
$$= \langle e_2(k_1, \omega_1)e_4(k_2, \omega_2)\rangle\langle g(k_3, \omega_3)g(k_4, \omega_4)\rangle$$
$$+ \langle e_2(k_1, \omega_1)g(k_3, \omega_3)\rangle\langle e_4(k_2, \omega_2)g(k_4, \omega_4)\rangle$$
$$+ \langle e_2(k_1, \omega_1)g(k_4, \omega_4)\rangle\langle e_4(k_2,\omega_2)g(k_3, \omega_3)\rangle.$$

In the diagrams, the pairing of two random functions will be indicated by joining these with a broken line. Thus, to show that the expectation value is to be taken in a diagram, we join with broken lines pairs of line ends corresponding to g, e_n in all possible ways. Usually, many such pairings give the same contribution. To illustrate this we give the diagrammatic form of the equations for R_2 and A_2

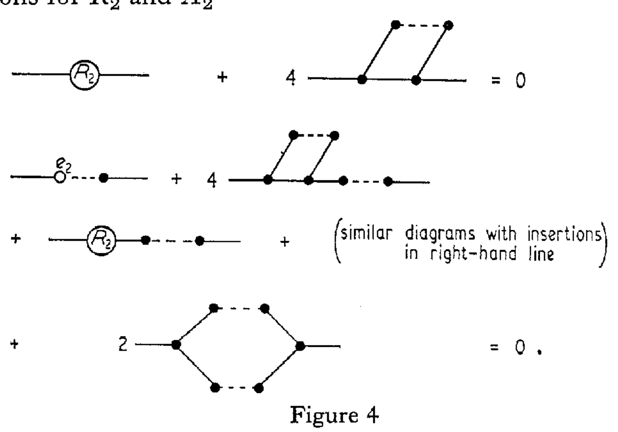

Figure 4

The second and third diagrams in the second equation cancel as a consequence of the first equation. Clearly, the broken line joining two g's gives a factor $VT\beta$ in the corresponding term.

To simplify the diagrams further and to show the correspondence with the formalism of Wyld we represent the two-velocity correlation function $U(k, \omega)$ by a wavy line, i.e. we replace

Figure 5

The equations may then be represented as follows

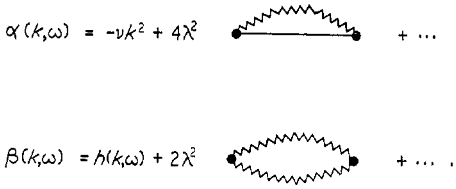

Figure 6

Equating to zero the coefficients of λ^4 in the series (5) and (6) enables one to write down the terms in λ^4 for α and β. The procedure is straightforward but rather tedious and is not reproduced here. The final result may be written simply in diagram form

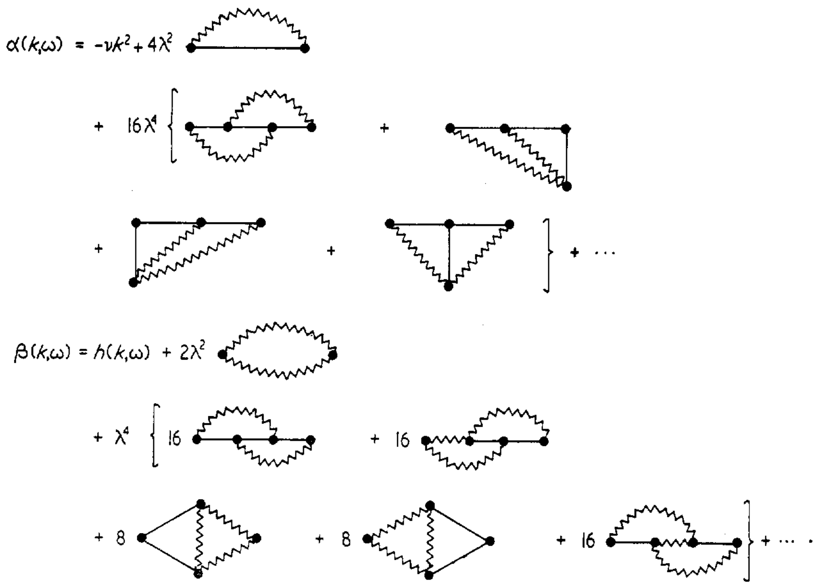

Figure 7

If we introduce the function $(i\omega + \nu k^2)^{-1}$ and represent it by a thin line, then the equations relating the response and correlation functions to α and β may be represented as follows:

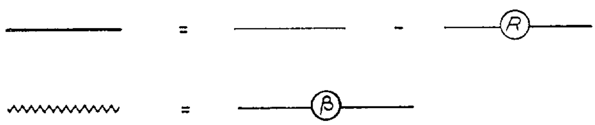

Figure 8

where R denotes $(\lambda^2 R_2 + \ldots)$.

It is interesting to note that, if one uses the perturbation series of Wyld to obtain expansions for α and β and retains only those diagrams which are irreducible in the sense that the internal lines contain no insertions, then these diagrams are just those appearing in figure 7 up to fourth order. Whether the higher-order diagrams of our series are also irreducible in this sense has not been ascertained. The diagrams are not irreducible in the usual sense since they may contain vertex corrections.

2.4. *Inclusion of the vertex function*

Finally, we indicate briefly how the procedure given above may easily be extended to give equations involving a 'dressed' vertex function as well as the response and correlation functions. To derive these we rewrite equation (1) in the form

$$\begin{aligned} i\omega v(k,\omega) &= \alpha(k,\omega)v(k,\omega) + g(k,\omega) + R(k,\omega)v(k,\omega) \\ &\quad + e(k,\omega) + \frac{\lambda}{VT}\sum K(k,\omega;k_1,\omega_1,k_2,\omega_2)v(k_1,\omega_1)v(k_2,\omega_2) \\ &\quad + \frac{\lambda}{VT}\sum L(k,\omega;k_1,\omega_1,k_2,\omega_2)v(k_1,\omega_1)v(k_2,\omega_2) \end{aligned}$$

where

$$\begin{aligned} R(k,\omega) &= \lambda^2 R_2(k,\omega) + \lambda^4 R_4(k,\omega) + \dots \\ e(k,\omega) &= \lambda^2 e_2(k,\omega) + \lambda^4 e_4(k,\omega) + \dots \\ L(k,\omega;k_1,\omega_1,k_2,\omega_2) &= \lambda^2 L_2(k,\omega;k_1,\omega_1,k_2,\omega_2) + \lambda^4 L_4(k,\omega;k_1,\omega_1,k_2,\omega_2) + \dots \end{aligned}$$

and

$$\begin{aligned} \alpha(k,\omega) + R(k,\omega) &= -\nu k^2 \\ g(k,\omega) + e(k,\omega) &= f(k,\omega) \\ K(k,\omega;k_1,\omega_1,k_2,\omega_2) + L(k,\omega;k_1,\omega_1,k_2,\omega_2) &= M(k,\omega;k_1,\omega_1,k_2,\omega_2). \end{aligned}$$

K and L are symmetric in k_1, ω_1 and k_2, ω_2 and conserve 'four-momentum'.

A perturbation series for $v(k,\omega)$ may be generated as before starting from the zeroth-order approximation $g(k,\omega)/\Omega(k,\omega)$. If the vertex K is represented by a small black square, then the corresponding diagrams are

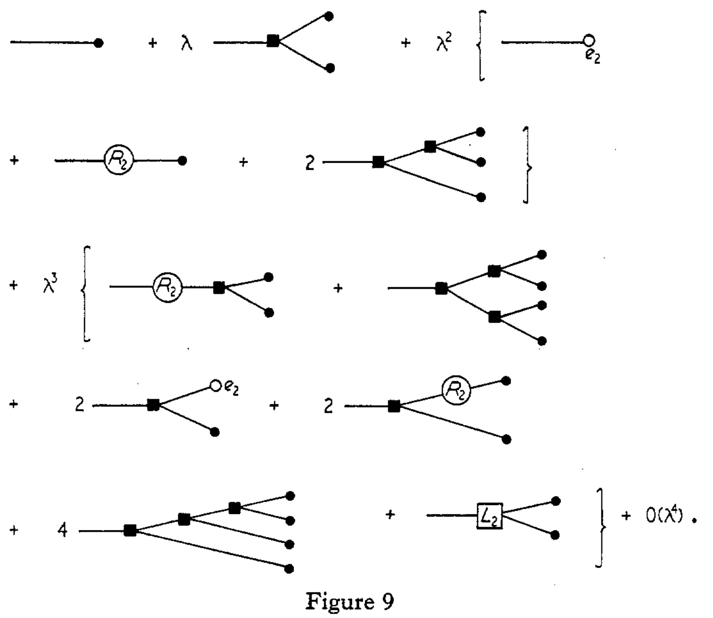

Figure 9

As before, we derive from this series the expansions for the response function and two-velocity correlation function and we impose the conditions that all except the zeroth-order terms of these series should vanish. In addition, we construct a series for the vertex function defined by

$$\frac{VT}{2\lambda}\left\langle\frac{\partial^2 v(k, \omega)}{\partial g(k_1, \omega_1)\partial g(k_2, \omega_2)}\right\rangle \Omega(k, \omega)\Omega(k_1, \omega_1)\Omega(k_2, \omega_2).$$

The corresponding diagrams are obtained from those of figure 9 by erasing two g's in all possible ways, together with the lines connecting them to the rest of the diagram, and averaging the remaining g's and e_n's. Also the line on the extreme left of the diagram is omitted.

It is seen that the zeroth-order term for this quantity is $K(k, \omega; k_1, \omega_1, k_2, \omega_2)$. We require, as usual, that all the higher-order terms are zero, thereby obtaining equations for L_2, L_4, etc. Up to order λ^4 the equations for α, β, and K are as follows:

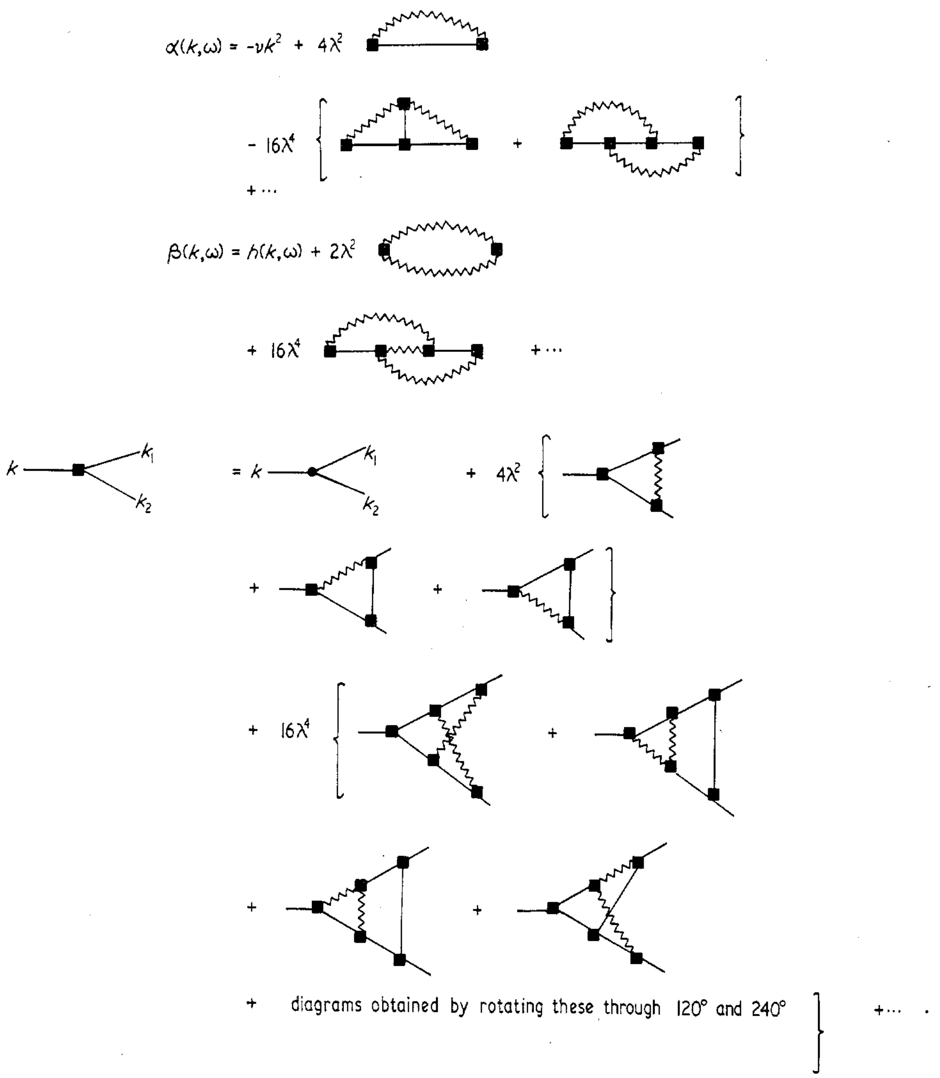

Figure 10

It will be observed that the equations for β and K are the same as those given by Wyld. The equation for α replaces the one for the second type of vertex in Wyld's theory. It is important to observe that the diagrams appearing in the series for α are not all irreducible (in the usual sense), for example, the first diagram of order λ^4 is not. This has the effect of avoiding the double counting of certain diagrams for the response function when the equations are iterated to generate an expansion in terms of the 'bare' response, correlation and vertex functions. It was for precisely this purpose that Wyld sought to replace the equation for the response function by one for a second sort of vertex from which it could be calculated by means of a 'Ward' identity. In the equation given by Lee, the response function is expressed in terms of diagrams which are irreducible but which contain the 'bare' vertex M in higher-order terms.

3. Conclusion

A self-consistent perturbation procedure has been demonstrated which leads directly to equations relating the response function and the two-velocity correlation function (and vertex function) without the necessity of analysing diagrams of arbitrarily high order. It differs from other theories based on a self-consistency procedure in yielding the direct-interaction approximation as the simplest non-trivial approximation.

The convergence properties of the series appearing in the final equations are completely unknown (the same is true of similar series appearing in quantum theory), and numerical solution of the equations for comparison with experiment has been limited to the direct-interaction approximation because of the complexity of terms beyond second order. Reasonable agreement with experiment has been found by Kraichnan; however, recently it has been found that experimental evidence seems to favour the Kolmogorov theory which gives a rather different spectral function. For details the papers of Kraichnan (1958, 1964) should be consulted.

It would be of interest to examine the relative magnitudes of the first few terms of the series for a simple system to which the above theory can be applied and for which the calculations involved can more easily be carried out. A suitable system might be, for example, the damped anharmonic oscillator with Gaussian forcing term.

Another problem is that of the realizability of approximations for the correlation function, i.e. the requirement that the approximation to the correlation function which emerges from some calculation scheme should have the properties of a correlation function. As yet, this has only been established for the direct-interaction approximation.

References

EDWARDS, S. F., 1964, *J. Fluid Mech.*, **18,** 239.
HERRING, J. R., 1965, *Phys. Fluids*, **8,** 2219.
KRAICHNAN, R., 1958, *J. Fluid Mech.*, **5,** 497.
—— 1964, *Phys. Fluids*, **7,** 1723.
LEE, L. L., 1965, *Ann. Phys., N.Y.* **32,** 292.
NOVIKOV, E. A., 1965, *Sov. Phys.-JETP*, **20,** 1290.
WYLD, H. W., JR., 1961, *Ann. Phys., N.Y.*, **14,** 143.